All calculations were performed by an IBM 360/67 with FORTRAN G compilation, and the computer code for the three methods was written by the same person.

Figure 1 is a three-dimensional representation of the results of this comparison, where the height above the m-n grid indicates the relative time required to obtain an $\bar{S}$ vector in CPU sec. Figure 2 is a quantitative presentation of the same data. The penalty in computation for using the simplex methods becomes even more apparent for larger problems. For example, with $m = n = 50$, Method IA required 160 sec and Method IB 80 sec, while Method II required only 15 sec.

Conclusions

We conclude that the quadratic Euclidian normalization is superior to the linear Cartesian normalization as a candidate for the length restriction (3) in the statement of the auxiliary problem. The method of solution using Norm II is clearly more efficient than the methods based on Norm I, and has the additional advantage of introducing no bias. It is particularly noteworthy that Method II run times are primarily a function of the number of active constraints only, while Methods IA and IB run times increase significantly with increase in either m or n.

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On Transient Cylindrical Surface Heat Flux Predicted from Interior Temperature Response

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I. Introduction

IN heat transfer studies surface temperature and heat flux are important quantities. However, many experimental difficulties arise in implanting a probe at the surface for heat transfer measurements; for example, involving the motion of a projectile over a barrel surface, sliding of a piston in the combustion chamber, melting or ablation of a heat shield, freezing or quenching of a material process, and high temperature exhaustion of a rocket engine. Furthermore, the presence of a probe at the surface disturbs the surface condition and flow process adjacent to it and thus the true heat transfer. Therefore, it is desirable in these cases that the prediction of surface temperature and heat flux be accomplished by inverting the temperature as measured by a probe located interior to the surface of the solid material.

In general, the preceding problem is known as the "Inverse Problem." Many configurations such as spheres, slabs, and cylinders have been studied, and many methods such as numerical, graphical, series, convolution integral, and Laplace transforms have been utilized. Stolz[1] and Beck[2] considered the numerical inversion of the integral solution for semi-infinite and spherical bodies. In this method, care is required in selecting a time interval in order to achieve a stable solution. Carslaw and Jaeger,[3] Burggraf,[4] Koveryanov,[5] and Shumakov,[6] respectively, considered different series approaches in which generally the local temperature and local heat flux at an interior location and their higher derivatives are required. Sparrow, Hadji-Sheikh, and Lundgren,[7] Imber and Kahn,[8] Imber,[9] Sabherwal,[10] Masket and Vastano,[11] and Deverall and Channapragada[12] applied the transform method. In these works, the solution is represented in either an integral form after some manipulation of the contour integral from the inverse transform, or as a series form after an expansion of the solution for small and large times. Beck,[13] in a series of papers, applied a finite-difference approximation in conjunction with a least-squares fit procedure as well as a nonlinear estimate method for the inverse heat conduction problem.

This paper reports a simple method of determining a short time transient surface temperature and heat flux for the case of a hollow cylinder based on the inversion of the temperature profile measured by only one interior probe.

II. Analysis

Consider a long hollow cylinder with sufficient wall thickness such that the outer surface temperature has a negligible response when the inner surface is exposed to a thermal pulse of a transient process. For example, in a gun barrel or combustion chamber the transient process requires a time duration in the order of milliseconds while the lag time of the outer surface for a significant response is in the order of seconds. This condition considerably simplifies the theoretical analysis as the outer boundary may be assumed to be infinite, and only one interior probe of the cylinder is required in the experimental measurement. The material of the cylinder is considered to be homogeneous and isotropic with constant thermal diffusivity, α. Let R_i and R_o be, respectively, the inner and outer surface radii, R_1 the radius of the probe location and t the dimensionless time. If the temperature of the cylinder is initially uniform at T_o, the mathematical problem governing the temperature T, may be written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} \qquad \begin{matrix} 1 < r < r_o \simeq \infty \\ t > 0 \end{matrix} \tag{1}$$

$$\theta(r, 0) = 0 \tag{2}$$

$$\theta(\infty, t) = 0 \tag{3}$$

$$\theta(r_1, t) = f(t) \qquad 1 < r_1 < \infty \tag{4}$$

where $\theta = T - T_o$, $r = R/R_i$, $t = \alpha\tau/R_i^2$, and $f(t)$ is the interior temperature response of the thermocouple measured at $r = r_1$ at the dimensionless time t. The problem is to predict surface temperature $\theta(1, t)$ and heat flux per unit area

$$q = -(K/R_i)(\partial\theta/\partial r)|_{r=1} \tag{5}$$

where K is the thermal conductivity.

The problem can be solved by Laplace transformation. Let the transformation be

$$\bar{\theta}(r, s) = \int_0^\infty \theta\, e^{-ts}\, dt \tag{6}$$

When θ satisfies the Dirichlet's condition the temperature function θ is recovered by inversion of the Laplace transformation as

$$\theta(r, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{\theta}\, e^{st}\, ds \tag{7}$$

Received October 10, 1974.
Index categories: LV/M Aerodynamic Heating; Rocket Engine Testing; Heat Conduction.

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where c is a suitable positive value. Equations (1) and (2) under transformation (6) become

$$(d^2\bar{\theta}/dr^2)+(1/r)(d\bar{\theta}/dr) = s\bar{\theta}$$

which has a solution of the form

$$\bar{\theta} = AI_o(Pr)+BK_o(Pr) \tag{8}$$

where I_o and K_o are modified Bessel functions of the first and second kind with $P = (s)^{1/2}$. With the boundary conditions (3) and (4), Eq. (8) becomes

$$\bar{\theta} = \bar{f}(s)[K_o(Pr)/K_o(Pr_1)] \tag{9}$$

where $\bar{f}(s)$ is the Laplace transform of the boundary condition (4).

The temperature response measured at $r = r_1$ can be expressed by a polynomial or numerous other suitable functions. In the present analysis, for reasons to be explained later, $f(t)$ will be represented by an N term polynomial with each term containing an integral of error function as follows

$$f(t) = \sum_{n=1}^{N} b_n(4t)^n\Gamma(n+1)i^{2n}\,\text{erfc}\left(\frac{r_1-1}{(4t)^{1/2}}\right) \tag{10}$$

where the b_n's are coefficients to be determined such that the N term series represents the function $f(t)$. The temperature solution on the Laplace plane which satisfies boundary condition (10) is then

$$\bar{\theta}(r,s) = \sum_{n=1}^{N} \frac{b_n\Gamma(n+1)}{s^{(1+n)}}e^{-(r_1-1)P}\frac{K_o(Pr)}{K_o(Pr_1)} \tag{11}$$

The temperature solution is then obtained by substituting Eq. (11) into Eq. (7). The general inversion of the solution is difficult, but for small time t (i.e. large s or P) and r not very small, we can expand asymptotically the modified Bessel function $K_o(Pr)/K_o(Pr_1)$ for large values of P (see Ref. 14) as

$$\frac{K_o(Pr)}{K_o(Pr_1)} = \left(\frac{r_1}{r}\right)^{1/2}\exp[-P(r-r_1)]\left[1+\frac{r-r_1}{8r_1rP}+\frac{9r_1^2-2rr_1-7r^2}{128r^2r_1^2p^2}+\frac{-150r_1^3+18r_1^2r+14r_1r^2+118r^3}{2048r_1^3r^3p^3}+\cdots\right] \tag{12}$$

Thus we have for a short time temperature response (temperature as measured at R_1) at surface $r = 1$,

$$\bar{\theta}_i = \sum_{N=1}^{N} b_n\frac{\Gamma(n+1)(r_1)^{1/2}}{s^{(1+n)}}\left[1+\frac{1-r}{8r_1p}+\frac{9r_1^2-2r_1^2-7}{128r_1^2p^2}+\frac{-150r_1^3+18r_1^2+14r_1+118}{2048r_1^3p^3}+\cdots\right] \tag{13}$$

It is noted that the exponential term in Eq. (12), when evaluated at $r = 1$, cancels the exponential term of Eq. (11), which explains the use of the polynomial involving error function in Eq. (10) for the representation of the experimental measured temperature data at location R_1 of the hollow cylinder. Inversion of Eq. (13) gives the inner surface temperature as

$$\theta_i = \sum_{n=1}^{N} b_nG_n(r_1,t) \tag{14}$$

with

$$G_n(r_1,t) = (r_1)^{1/2}t^n\Gamma(n+1)\left[\frac{1}{\Gamma(n+1)}+\frac{(1-r_1)t^{1/2}}{8r_1\Gamma(n+\frac{3}{2})}+\frac{(9r_1^2-2r_1-7)t}{128r_1^2\Gamma(n+2)}+\frac{(-150r_1^2+18r_1^2+14r_1+118)t^{3/2}}{2048r_1^3\Gamma(n+\frac{5}{2})}\right] \tag{15}$$

$G_n(r_1,t)$ is a function of time and probe location only and is independent of heat flux or surface temperature for a given problem.

Surface heat flux may be obtained by differentiating Eq. (11) with respect to R coupled with Eq. (12) as, after inversion

$$q = \frac{-K}{R_i}\sum_{n=1}^{N} b_nH_n(r_1,t) \tag{16}$$

where

$$H_n(r_1,t) = -\tfrac{1}{2}G_n(r_1,t)-(r_1)^{1/2}\Gamma(n+1)t^n\left[\frac{t^{-1/2}}{\Gamma(n+\frac{1}{2})}+\frac{1-r_1}{8r_1\Gamma(n+1)}+\frac{(-7r_1^2-2r_1-7)t^{1/2}}{128r_1^2\Gamma(n+\frac{3}{2})}+\frac{(138r_1^3+14r_1^2+14r_1+118)t}{2048r_1^3\Gamma(n+2)}+\frac{(450r_1^2+36r_1+14)t^{3/2}}{2048r_1^2\Gamma(n+\frac{5}{2})}+\cdots\right] \tag{17}$$

Similarly, $H_n(r_1,t)$ is a function of time and location of the thermal probe only. Thus $H_n(r_1,t)$ and $G_n(r_1,t)$ can be considered as universal functions independent of specific heat flux at the surface. If the gas temperature, $T_g(t)$, in the hollow cylinder core is known, then the instantaneous heat transfer coefficient, $h(t)$ can be determined where heat flux $q(t)$ and wall temperature $T_w(t)$ (or θ_i+T_o) is given by Eqs. (16) and (14), respectively. That is

$$h(t) = [q(t)/T_g(t)-T_w(t)] \tag{18}$$

and the average heat transfer coefficient $\bar{h}(t)$ can then be defined as

$$\bar{h}(t) = \frac{\int_0^t q(t^1)\,dt^1}{\int_0^t [T_g(t^1)-T_w(t^1)]\,dt^1} \tag{19}$$

III. Discussion

The solution obtained above is restricted to a small time duration because the modified Bessel function in Eq. (11) was expanded asymptotically. It is therefore desirable to have an estimate of the time duration in which the solution is valid. The asymptotic expansion of the Bessel function $K_o(Pr)/K_o(Pr_1)$ in Eq. (12) is given by Watson[14] to be valid for the argument Pr greater than 10. Consequently, substituting $(s)^{1/2} = 1$ for $Pr > 10$, a conservative estimate of the real time interval for a valid solution is $\tau < R_i^2/100\alpha$. For example, a temperature probe located at an interior point, $r_1 = 1.05$ in. in a steel hollow cylinder with a radius of 0.5 in. ($\alpha = 0.04$ ft^2/hr), the solution is valid for a time interval of 250 msec which is adequate for gun barrel problems. On the other hand, with the time duration of 250 msec, the depth for a sensible penetration of temperature, for example at which the temperature is approximately less by an exponential factor from the surface temperature, is $\Delta R = (\alpha\tau)^{1/2} = 0.05$ in. in this case. Thus, this solution would be valid for predicting transient surface temperatures and inner surface heat fluxes for the steel hollow cylinder with a wall thickness greater than 0.1 in. It is apparent that the solution is valid for an even longer period of time if the radius of cylinder is large such as automobile engines or rocket nozzles and if the thermal diffusivity of the material is small such as insulation material used in furnace and process chamber.

The computer program for the solution and several examples are given in the report,[15] where a maximum of nineteen temperature-time data inputs may be used to determine nineteen b_n coefficients of the polynomial function. It should be noted that in order to predict surface heat fluxes and temperatures for short times a sufficient duration of thermocouple response is required. For example, to accurately predict the bore temperatures and heat fluxes of the above steel cylinder for 50 msec based on a thermocouple located 0.050 in. from the bore, a 2 sec thermocouple response would be required. The long response time is due to the diffusional behavior of the transient heat conduction which requires time to diffuse the heat flux input at the surface to the thermocouple.

The advantage of the present method is that it requires only the measurement of temperature response at one interior location while the previous works require either measurement at two locations or at one location but with additional gradient quantities which are difficult to make accurate measurements.

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Combustion of Mixed Fuels

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Introduction

STUDIES on the ignition characteristics and performance parameters of a liquid propellant system having nitric acid as an oxidizer and the mixture of Hydrazine + ethyl alcohol as mixed fuel have been reported earlier.[1] To have a more comprehensive picture, additional and supplementary combustion studies on the system were undertaken particularly in view of the fact that the mixed fuel can be used in external combustion engines like Stirling engine. These studies are reported in the present communication.

Experimental

Materials

Anhydrous hydrazine was prepared by dehydrating hydrazine hydrate as described earlier.[2] Ethyl alcohol was prepared by purifying absolute alcohol in the usual manner.[3]

Received October 11, 1974. Our thanks are due to the Ministry of Defence and Council of Scientific and Industrial Research for supporting the investigation.

Index categories: Fuels and Propellants, Properties of; Liquid Rocket Engines; Combustion Stability, Ignition and Detonation.

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1) *Measurement of mass burning rate:* Mass burning rate was determined by a specially designed burner which consisted of two glass tubes *A* and *B*, the diameter of *A* being 1 cm. The tubes *A* and *B* were connected by polythene tubing. *B* was used for burning, and the diameter of *B* could be varied. *A* had a bent portion in the upper part and a side-tube attached with a burette;

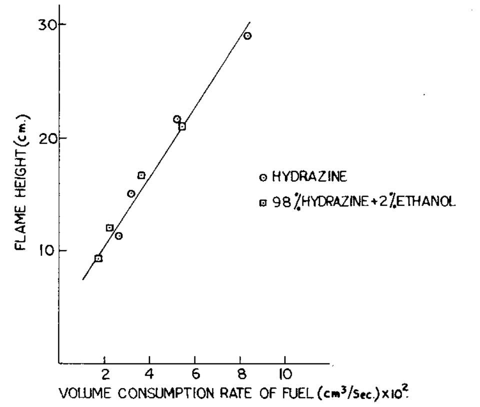

Fig. 1 Test of Eq. (3)

the side tube was maintained at the same level as the tip of *B*. Fuel could be introduced into *A* with a known rate, and the excess of the fuel overflowing through the side-tube could be measured. The tip of *B* was ignited, and the amount of fuel consumed in air at different time intervals could be measured.

Burning rate studies were also made in N_2O_4/NO_2 oxidizing atmosphere by enclosing the burner with a Pyrex glass tube (6 cm. i.d.) which was flushed with NO_2. NO_2 was prepared by the usual laboratory method.[4] The results were plotted against time. A linear plot was obtained from which the mass burning rate was calculated. The data are given in Table 1.

2. *Measurement of flame height:* Flame height was measured visually with the help of a metallic scale placed behind the flame. A number of measurements were made during each run and the average values of the flame heights are plotted against volume consumption rate as shown in Fig. 1.

3) *Measurement of temperature profile:* Temperature of the flame at various points was recorded potentiometrically with the help of Chromel-Alumel thermocouple having bead diameter ~1.5 mm the cold junction of which was placed in melting ice. For measuring the temperature distribution in the liquid phase a modification was made in the burner design so as to allow the thermocouple junction to be moved up and down inside the tube. Results are plotted in Fig. 2.

III. Discussion

According to Spalding the rate of pool burning and vessel diameter are related as follows.[5]

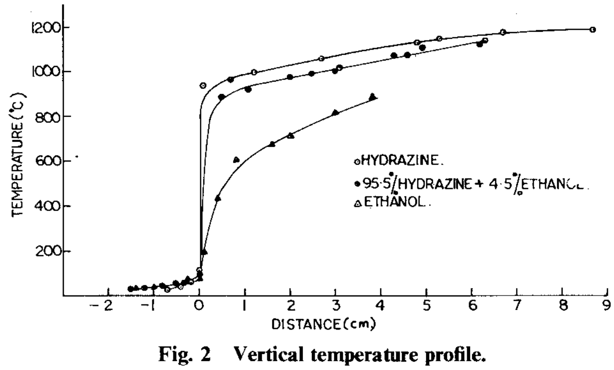

Fig. 2 Vertical temperature profile.